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On the eigenvalues of $S \cdot \pi$ for arbitrary spin in a constant magnetic field

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Abstract. Utilising the intimate connection of a charged particle in a homogeneous magnetic field to that of a harmonic oscillator, we demonstrate explicitly that the eigenvalue spectrum for the matrix operator $S \cdot \pi$ for arbitrary spin in a uniform magnetic field in the z direction is governed, for its discrete part, by that of a Hermitian matrix defined on the space of the particle number and spin operators and is thus constrained to be real for any intensity of the external magnetic field. The present analysis generalises to arbitrary spin our previously reported results applied to the case of a spin-1 particle.

1. Introduction

In an earlier comment (Jayaraman and de Oliveira 1985) we demonstrated that the eigenvalue spectrum for spin 1 of the matrix operator $S \cdot \pi$ where S_i $(i = 1, 2, 3)$ are the spin matrices, $\pi = -i\nabla - eA$ the generalised momentum (we set $c = \hbar = 1$), e the charge and $\{A_0, A\} = \{0, \frac{1}{2}H(-y, x, 0)\}$ the 4-vector potential for a constant magnetic field H in the z direction, is purely real for any intensity of H, thereby removing a misconception existing in the literature (Weaver 1978) that some of the eigenvalues can indeed become complex for sufficiently intense H. Here we extend the results to the case of arbitrary spin, demonstrating that the eigenvalue spectrum of $S \cdot \pi$ is purely real for any spin and for any intensity of H.

Transforming the problem for arbitrary spin to that of a harmonic oscillator (Mathews 1974, Mathews and Venkatesan 1986) we demonstrate explicitly in § 2 in a parallel way to that presented earlier for the spin-1 case (Jayaraman and de Oliveira 1985) that such a conversion leads to a Hermitian matrix defined on the space of the particle number and spin operators which governs the discrete part of the spectrum for any intensity of the external magnetic field, thus constraining the eigenvalues to be purely real. Though the transcription of our general results here to the case of spin-$\frac{3}{2}$ can be handled directly and compared with those of Weaver's analysis for spin-$\frac{3}{2}$ (Weaver 1978), and following analogous reasons explained by us in detail for the spin-1 case (Jayaraman and de Oliveira 1985) the incorrectness of the latter's results admitting some complex eigenvalues for sufficiently intense H established, we skip this comparison here. In § 3 we present a brief discussion of the implications of our results on the energy eigenvalue spectrum associated with arbitrary spin Hamiltonians in relativistic theories, interacting with a constant magnetic field.

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2. The nature of the eigenvalue spectrum of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for any spin

We start with the proposed eigenvalue problem

$$\boldsymbol{S}\cdot\boldsymbol{\pi}\psi(x,y,z)=\lambda_s\psi(x,y,z) \tag{1}$$

with $\boldsymbol{S}=(S_1,S_2,S_3)$ being the conventional spin-s angular momentum matrices. Noting that the operator $\pi_3=-\mathrm{i}\partial/\partial z$ commutes with everything in $\boldsymbol{S}\cdot\boldsymbol{\pi}$ we set $\psi(x,y,z)=\phi(x,y)\exp(\mathrm{i}p_3z)(-\infty<p_3<\infty)$ in (1) so that $\phi(x,y)$ satisfies

$$[\tfrac{1}{2}(S_+a^++S_-a^-)+S_3a_3]\phi(x,y)=\bar{\lambda}_s\phi(x,y) \qquad \bar{\lambda}_s=(2eH)^{-1/2}\lambda_s \tag{2}$$

where $S_\pm=S_1\pm\mathrm{i}S_2$ and

$$a^\mp=(2eH)^{-1/2}\pi_\pm=(2eH)^{-1/2}(\pi_1\pm\mathrm{i}\pi_2)=(a^\pm)^\dagger \tag{3}$$

with $a_3=(2eH)^{-1/2}p_3$ being any fixed real number. The operators $a^\pm$ together with the number operator N_a defined by

$$N_a=a^+a^-=(2eH)^{-1}(\pi_1^2+\pi_2^2)-\tfrac{1}{2} \qquad \pi^2=\pi_1^2+\pi_2^2+\pi_3^2 \tag{4}$$

satisfy the simple harmonic oscillator algebra

$$[a^-,a^+]_-=1 \qquad [N_a,a^\mp]_-=\mp a^\mp. \tag{5}$$

In terms of the mutually commuting simple harmonic oscillator operators in the x and y directions, defined by

$$a_i^\mp=\frac{1}{\sqrt{2}}\left(\pm\frac{\partial}{\partial\bar{x}_i}+\bar{x}_i\right) \qquad N_i=a_i^+a_i^- \qquad [N_i,a_i^\mp]_-=\mp a_i^\mp \qquad (i=1,2) \tag{6}$$

$$x_1=x=\left(\frac{2}{eH}\right)^{1/2}\bar{x} \qquad x_2=y=\left(\frac{2}{eH}\right)^{1/2}\bar{y} \qquad a_{1,2}^\mp=a_{x,y}^\mp \qquad N_{1,2}=N_{x,y} \tag{7}$$

the operators $a^\mp$ and N_a of equations (3)-(5) take their explicit forms:

$$a^\mp=\frac{1}{\sqrt{2}}(a_y^\mp\mp\mathrm{i}a_x^\mp) \qquad N_a=\tfrac{1}{2}(N_x+N_y-L_z) \tag{8}$$

where

$$L_z=-\mathrm{i}(a_x^+a_y^--a_y^+a_x^-)=-\mathrm{i}x\frac{\partial}{\partial y}+\mathrm{i}y\frac{\partial}{\partial x}. \tag{9}$$

Apart from the spin degrees of freedom involved in (2), the operators $a^\mp$ effectively carry, in view of the first of the equations in (8), just one degree of freedom though the wavefunction $\phi(x,y)$ contains obviously two degrees of freedom. However, defining a second set of harmonic oscillator operators $b^\mp$ by

$$b^\mp=(2eH)^{-1/2}\pi_\pm^*=\frac{1}{\sqrt{2}}(a_y^\mp\pm\mathrm{i}a_x^\mp) \tag{10a}$$

$$[b^-,b^+]_-=1 \qquad [N_b,b^\mp]_-=\mp b^\mp \tag{10b}$$

$$N_b=b^+b^-=\tfrac{1}{2}(N_x+N_y+L_z) \tag{10c}$$

so that (8) and (10a) constitute a unitary transformation from the old set $(a_x^\mp,a_y^\mp)$ to the new set $(a^\mp,b^\mp)$, it is evident that $b^\mp$ together with $a^\mp$ of (2) complete the total of two degrees of freedom. Also, just as the old set of operators $(a_x^\mp,N_x)$ and $(a_y^\mp,N_y)$ commute mutually so do the new sets $(a^\mp,N_a)$ and $(b^\mp,N_b)$ as can be directly verified.

Defining now a complete set of orthonormal states in the space of the mutually commuting set of operators $N_a(\to n_a = n)$, $N_b(\to n_b)$ and $S^2 = S_1^2 + S_2^2 + S_3^2(\to s(s+1))$ by

$$|n, \alpha_i; n_b\rangle = |n\rangle \otimes |\alpha_i\rangle \otimes |n_b\rangle$$
$$\langle n'\alpha_j; n'_b | n\alpha_i; n_b\rangle = \delta_{nn'}\delta_{\alpha_j\alpha_i}\delta_{n'_b n_b} \tag{11a}$$

$$n, n', n_b, n'_b = 0, 1, 2, \ldots; \alpha_i, \alpha_j = s, \ldots, -s \tag{11b}$$

it is evident that the number eigenvalue n in

$$N_a|n\alpha_i; n_b\rangle = n|n\alpha_i; n_b\rangle \qquad n = 0, 1, 2, \ldots \tag{12}$$

is infinitely degenerate as n does not depend on n_b. Also, since the operators $b^\mp$ of (10a) simply commute with everything in $\boldsymbol{S}\cdot\boldsymbol{\pi}$ of (1), i.e.

$$[b^\mp, \boldsymbol{S}\cdot\boldsymbol{\pi}]_- = 0 \tag{13}$$

it is readily inferred that, in the expansion of $\phi(x, y)$ of (2) in terms of the complete set of basis states $|n\alpha_i\rangle \otimes |n_b\rangle$, the operators $a^\mp$ and S_i in (2) do not affect the quantum number n_b at all. Hence $\bar{\lambda}_s$ of (2) is infinitely degenerate with n_b taking the infinite set of values 0, 1, 2, ... For an explicit construction of the Schrödinger wavefunctions for the simultaneous eigenkets of N_a and $L_z = N_b - N_a$ (which equality is a simple consequence of (8) and (10c)) the reader is referred to the elegant work of Mathews (Mathews 1974, Mathews and Venkatesan 1986). In the rest of our analysis here we will suppress the label n_b in $|n\alpha_i; n_b\rangle$ as it serves no further role than endowing infinite degeneracy for $\bar{\lambda}_s$ of (2) as explained above.

Now, the following properties of $|n, \alpha_i\rangle$ follow readily:

$$N_a|n, \alpha_i\rangle = n|n, \alpha_i\rangle \tag{14a}$$

$$a^-|0, \alpha_i\rangle = 0 \tag{14b}$$

$$|n, \alpha_i\rangle = \frac{1}{(n!)^{1/2}}(a^+)^n|0, \alpha_i\rangle \tag{14c}$$

$$a^-|n, \alpha_i\rangle = \sqrt{n}\,|n-1, \alpha_i\rangle \tag{14d}$$

$$a^+|n, \alpha_i\rangle = (n+1)^{1/2}|n+1, \alpha_i\rangle \tag{14e}$$

$$S_3|n, \alpha_i\rangle = \alpha_i|n, \alpha_i\rangle \tag{15}$$

$$\begin{aligned}
&S_+|n, s\rangle = 0 \qquad S_+|n, s-1\rangle = (2s\cdot 1)^{1/2}|n, s\rangle \\
&S_+|n, s-2\rangle = [(2s-1)\cdot 2]^{1/2}|n, s-1\rangle \\
&S_+|n, s-3\rangle = [(2s-2)\cdot 3]^{1/2}|n, s-2\rangle \qquad \ldots \\
&S_+|n, -s+3\rangle = [(2s-3)\cdot 4]^{1/2}|n, -s+4\rangle \qquad S_+|n, -s+2\rangle = [(2s-2)\cdot 3]^{1/2}|n, -s+3\rangle \\
&S_+|n, -s+1\rangle = [(2s-1)\cdot 2]^{1/2}|n, -s+2\rangle \qquad S_+|n, -s\rangle = (2s\cdot 1)^{1/2}|n, -s+1\rangle
\end{aligned} \tag{16}$$

$$\begin{aligned}
&S_-|n, s\rangle = (2s\cdot 1)^{1/2}|n, s-1\rangle \qquad S_-|n, s-1\rangle = [(2s-1)\cdot 2]^{1/2}|n, s-2\rangle \\
&S_-|n, s-2\rangle = [(2s-2)\cdot 3]^{1/2}|n, s-3\rangle \\
&S_-|n, s-3\rangle = [(2s-3)\cdot 4]^{1/2}|n, s-4\rangle \qquad \ldots \\
&S_-|n, -s+3\rangle = [(2s-2)\cdot 3]^{1/2}|n, -s+2\rangle \qquad S_-|n, -s+2\rangle = [(2s-1)\cdot 2]^{1/2}|n, -s+1\rangle \\
&S_-|n, -s+1\rangle = (2s\cdot 1)^{1/2}|n, -s\rangle \qquad S_-|n, -s\rangle = 0
\end{aligned} \tag{17}$$

$$\langle n', \alpha_j|n, \alpha_i\rangle = \delta_{n'n}\delta_{\alpha_j\alpha_i} \qquad n', n = 0, 1, 2, \ldots; \alpha_i, \alpha_j = s, \ldots, -s. \tag{18}$$

Expanding $\phi(x, y)$ of (2) in terms of the basis $|n, \alpha_i\rangle$, we have the expansion

$$\phi(x, y) = \sum_{\alpha_i = s}^{-s} \left(\sum_{n=0}^{\infty} c_{n\alpha_i} |n, \alpha_i\rangle \right) \tag{19}$$

where $c_{n\alpha_i}$ are the expansion coefficients. Since $c_{n\alpha_i}$ for $n<0$ do not find a place in (19) we have that

$$c_{n\alpha_i} = 0 \qquad n < 0. \tag{20}$$

Now it follows directly that $\boldsymbol{S} \cdot \boldsymbol{\pi}$ commutes with $N - S_3$ but not with N and S_3 separately, though N and S_3 commute among themselves (see the commutation relations (27a-d) in Jayaraman and de Oliveira (1985) together with equation (4) here). This means that the eigenvectors $\phi(x, y)$ in (2) can be labelled by the eigenvalues of $N - S_3$, i.e. by the values of $n - \alpha_i$ which implies that $c_{n\alpha_i}$ of (19) should occur for the same values of '$n - \alpha_i$' for $\phi(x, y)$ of (2). That this turns out to be the actual case follows straight away by substituting (19) in (2) and making use of the properties (14)-(18). In fact we obtain, after a considerable simplification, a set of algebraic equations for the $(2s+1)$ coefficients c_{ns}, $c_{(n-1)(s-1)}$, $c_{(n-2)(s-2)}, \ldots, c_{(n-2s+1)(-s+1)}$ and $c_{(n-2s)(-s)}$ (all with the same value $n-s$ of '$n - \alpha_i$') for all $n = 2s, 2s+1, 2s+2, \ldots$, etc, which can be written in the form of the following eigenvalue equation of a $(2s+1) \times (2s+1)$ matrix A for the column vector $c = (c_{ns}, c_{(n-1)(s-1)}, c_{(n-2)(s-2)}, \ldots, c_{(n-2s+1)(-s+1)}, c_{(n-2s)(-s)})$:

$$Ac = \bar{\lambda}_s c \tag{21}$$

and the matrix A is given explicitly by

$$A = \begin{bmatrix} sa_3 & \frac{\sqrt{2sn}}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\sqrt{2sn}}{2} & (s-1)a_3 & \frac{\sqrt{(2s-1)2(n-1)}}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\sqrt{(2s-1)2(n-1)}}{2} & (s-2)a_3 & \frac{\sqrt{(2s-2)3(n-2)}}{2} & \cdots & 0 & 0 \\ 0 & 0 & \frac{\sqrt{(2s-2)3(n-2)}}{2} & (s-3)a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-s+1)a_3 & \frac{\sqrt{2s(n-2s+1)}}{2} \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{2s(n-2s+1)}}{2} & -sa_3 \end{bmatrix}$$

$$n = 2s, 2s+1, \ldots. \tag{22}$$

As A is Hermitian (in fact, real and symmetric) its eigenvalues $\bar{\lambda}_s$ are real for any intensity of the external magnetic field H and for all $n = 2s, 2s+1, \ldots$, etc, and can be obtained by the resolution of the characteristic equation $|A - \bar{\lambda}_s I| = 0$ but is not attempted here.

However, the cases $n=0$, $n=1, \ldots, n = 2s-2$ and $n = 2s-1$ occur as special ones and the effective dimensionality of the matrix A in these cases is respectively one, two, three, $\ldots$, $(2s-1)$ and $2s$ by virtue of (20). We now note the important point that, in all these exceptional cases, the matrix A is also Hermitian and hence the eigenvalues $\bar{\lambda}_s$ are constrained to be real for any intensity of H.

The above completes the proof of our assertion that the spectrum of eigenvalues of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ is purely real for all $n=0,1,2,\ldots$, and for any intensity of the external magnetic field H.

3. Discussion

Our work here establishing the reality of the eigenvalue spectrum of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for arbitrary spin in a constant magnetic field has served to remove the misconception existing in the recent literature (Weaver (1978) for $s=1$ and $s=\frac{3}{2}$) that this spectrum also includes, for some values of $n\leqslant 2s-1$, complex values for sufficiently intense magnetic fields. The pure reality of the spectrum of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ demonstrated here, though for a constant magnetic field, is gratifying to start with in its beneficial implications on the nature of eigenvalues of higher-spin relativistic Hamiltonians. Considering, for example, relativistic wave equations for unique spin s in the Schrödinger form $\mathrm{i}(\partial/\partial t)\Psi=\mathscr{H}\Psi$ in interaction with a constant magnetic field in the z direction, the choice of a specific anomalous coupling dictated by the commutativity of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ with the operator $O=\pi^2-2eS_3H$ or, which is the same, with N_a-S_3 in view of (4) such that the Hamiltonian $\mathscr{H}$ (which is a function depending on $\boldsymbol{S}\cdot\boldsymbol{\pi}$ and O) commutes with $\boldsymbol{S}\cdot\boldsymbol{\pi}$ and O, it is clear from the present work that the contribution to the energy eigenvalues from the factors of $\boldsymbol{S}\cdot\boldsymbol{\pi}$, and also O in $\mathscr{H}$ can only be real. Under these conditions, if the energy spectrum could still include complex values for sufficiently intense H, this may be due to the specific construction of the Hamiltonian not being Hermitian in the ordinary sense but only with respect to a relativistically invariant scalar product involving a metric operator as, for example, in the case of the Sakata–Taketani (Taketani and Sakata 1940) Hamiltonian $\mathscr{H}_{\mathrm{ST}}$ for spin 1 treated by Weaver (1976) with the choice of a specific anomalous coupling with a constant magnetic field such that the mutually commuting $\boldsymbol{S}\cdot\boldsymbol{\pi}$ and O also commute with $\mathscr{H}_{\mathrm{ST}}$, a function of the former operators. It is of considerable interest to apply the direct procedure of this paper to reinvestigate the precise nature of the energy eigenvalue spectrum of $\mathscr{H}_{\mathrm{ST}}$ for that particular anomalous coupling of Weaver (1976) with a constant magnetic field and, as well, varying the anomalous coupling parameter with the objective of possibly obtaining a causal coupling, i.e. a specific anomalous coupling for which the energy spectrum is purely real (see, for example, Mathews (1974) for the spin-1 Proca theory and Prabhakaran and Seetharaman (1973) for the spin-1 Shay–Good (Shay and Good 1969) theory) and still possibly extend the notion of a causal coupling to different unique spin-$\frac{3}{2}$ theories in the Schrödinger form $\mathrm{i}(\partial/\partial t)\Psi=\mathscr{H}\Psi$, as, for example, of Weaver *et al* (1964), Mathews (1966a, b), Moldauer and Case (1956) and Guertin (1974), in interaction with a constant magnetic field. The results of our investigations on these questions will be reported separately.

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